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# Realizations of *ADE* type logarithmic principal *W*-algebras

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## 1 Introduction

**Definition 1.** Let  $V$  be a vertex operator algebra (VOA) and the vertex operator of  $a \in V$  is denoted by

$$Y(a, z) = a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \in \text{End } V[[z^{\pm}]].$$

1. The VOA  $V$  is  $C_2$ -cofinite if  $\dim R_V < \infty$ . Here,  $R_V = V/C_2(V)$  and

$$C_2(V) = \text{span}_{\mathbb{C}}\{a_{-2}b \mid a, b \in V\}.$$

2. The VOA  $V$  is rational if the representation category  $\text{Rep } V$  of  $V$  is semisimple, i.e.  $\forall V$ -module  $M \in \text{Rep } V$  is decomposed into the direct sum of irreducible  $V$ -modules.

The theory of  $C_2$ -cofinite BUT irrational VOAs is less complete than that of  $C_2$ -cofinite and rational VOAs, because only a few examples of  $C_2$ -cofinite but irrational VOAs are found. Our aim is to construct many  $C_2$ -cofinite but irrational VOAs, and the strategy is to generalize the well-known example of such kinds of VOAs. One of the most well-known examples of  $C_2$ -cofinite but irrational VOAs is the triplet  $W$ -algebra. This VOA is defined as the kernel of the narrow screening operator on the rescaled root lattice of  $A_1$  type, and studied by many people (e.g. [FGST1]-[FGST3], [AM1]-[AM3], [NT], [TW]). The definition of the triplet  $W$ -algebra is generalized in *ADE* types immediately, and we call these VOAs the logarithmic principal  $W$ -algebras (In this report, we omit “principal” and call these VOAs logarithmic  $W$ -algebras for simplicity). However, despite of the importance of logarithmic  $W$ -algebras in studies of  $C_2$ -cofinite but irrational VOAs, there are not much known except for the case of  $A_1$  type (triplet  $W$ -algebras) because of the complicated structures.

On the other hands, B.L.Feigin and I.Yu.Tipunin introduced a geometric approach to the studies of logarithmic  $W$ -algebras and their characters [FT]. They introduced Feigin-Tipunin algebras as sheaf cohomologies on the flag varieties and conjectured the following (we call Feigin-Tipunin conjecture):

1. The Feigin-Tipunin algebras are geometric realizations of the logarithmic  $W$ -algebras.
2. The character formulas of the logarithmic  $W$ -algebras.
3. The  $W$ -algebra module structures the logarithmic  $W$ -algebras.

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The author proved this conjecture in his master thesis. In this report, we want to introduce this result.

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## 2 Some results on the triplet $W$ -algebras

In order to avoid duplication, we give the setting in general  $ADE$  types all along.

### 2.1 Setting

Let  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$  be an  $ADE$  type simple Lie algebra of rank  $l$  and its triangular decomposition,  $\mathfrak{h}$  and  $\mathfrak{b} = \mathfrak{n}_- \oplus \mathfrak{h}$  its Cartan and Borel subalgebras respectively,  $G$ ,  $H$ , and  $B$  the semisimple, simply-connected, complex algebraic groups corresponding to  $\mathfrak{g}$ ,  $\mathfrak{h}$ ,  $\mathfrak{b}$  respectively. The labelling of the Dynkin diagrams are the one in [B]. Let  $Q$  be the root lattice of  $\mathfrak{g}$ ,  $Q'$  the weight lattice of  $\mathfrak{g}$ ,  $Q'_+$  the set of dominant integral weights of  $\mathfrak{g}$ ,  $\alpha_1, \dots, \alpha_l$  the simple roots of  $\mathfrak{g}$ ,  $\Pi$  the set of simple roots of  $\mathfrak{g}$ ,  $\omega_1, \dots, \omega_l$  the fundamental weights of  $\mathfrak{g}$ . We denote by  $(\cdot, \cdot)$  the standard invariant form of  $\mathfrak{g}$ ,  $W$  the Weyl group of  $\mathfrak{g}$ ,  $(c_{ij})$  the Cartan matrix of  $\mathfrak{g}$  and  $(c^{ij})$  the inverse matrix to  $(c_{ij})$ ,  $\rho$  the half sum of positive roots,  $h$  the (dual) Coxeter number of  $\mathfrak{g}$ ,  $\Omega$  the abelian group  $Q'/Q$ . We choose the representatives of elements from  $\Omega$  in  $Q'$  in the following way: for  $A_l$ ,  $D_l$ ,  $E_6$ ,  $E_7$ ,  $E_8$ , we choose  $\{0, \omega_1, \dots, \omega_l\}$ ,  $\{0, \omega_1, \omega_{l-1}, \omega_l\}$ ,  $\{0, \omega_1, \omega_3\}$ ,  $\{0, \omega_2\}$ ,  $\{0\}$  respectively. For  $X = A$  or  $D$  or  $E$ ,  $X_l$  means that  $\mathfrak{g}$  is the  $X$  type simple Lie algebra of rank  $l$ . We fix an integer  $p \in \mathbb{Z}_{\geq 2}$ .

### 2.2 Lattice VOA and the irreducible modules

Let  $\mathcal{L}$  be the lattice VOA associated to the rescaled root lattice  $\sqrt{p}Q$ . In a manner of speaking, lattice VOAs is “lattice  $\otimes$  Fock spaces”. We explain this: Let  $\mathcal{F}_0$  be the rank  $l$  Heisenberg vertex algebra. By abuse of notation, we sometimes use  $\mathcal{F}_0$  as the rank  $l$  Heisenberg algebra. By using this notation, we denote by  $\mathcal{F}_\alpha = \mathcal{F}_0|\alpha\rangle$  the Fock space corresponding to  $\alpha$ . Then, as a vector space,

$$\mathcal{L} \simeq \bigoplus_{\alpha \in \sqrt{p}Q} \mathcal{F}_\alpha \quad (1)$$

and the basis is given by

$$\{(\alpha_{i_1})_{-n_1} \cdots (\alpha_{i_k})_{-n_k} | \sqrt{p}\beta\rangle \mid 1 \leq i_1 \leq \cdots \leq i_k \leq l, \alpha_{i_j} \in \Pi, n_j \in \mathbb{N}, \beta \in Q\}. \quad (2)$$

Irreducible modules of  $\mathcal{L}$  are classified by elements of abelian group  $\Lambda = \frac{1}{\sqrt{p}}Q'/\sqrt{p}Q$  ([D]). We choose the basis elements  $\{\lambda_j = \frac{1}{\sqrt{p}}\omega_j \mid 1 \leq j \leq l\}$  of  $\frac{1}{\sqrt{p}}Q'$ . For each equivalence class  $\langle \lambda \rangle \in \Lambda$ , we choose a unique representative  $\lambda \in \frac{1}{\sqrt{p}}Q'$  of  $\langle \lambda \rangle \in \Lambda$  as

$$\lambda = -\sqrt{p}\hat{\lambda} + \bar{\lambda} = -\sqrt{p}\hat{\lambda} + \sum_{j=1}^l s_j \lambda_j, \quad (3)$$

where  $\hat{\lambda} \in Q'_+/Q \cap Q'_+$  and  $s_j = 0, \dots, p-1$ .

For  $\lambda \in \frac{1}{\sqrt{p}}Q'$ , we denote the irreducible module of  $\mathcal{L}$  corresponding to  $\langle \lambda \rangle \in \Lambda$  by

$$\mathcal{L}_{\langle \lambda \rangle} = \bigoplus_{\alpha \in \sqrt{p}Q} \mathcal{F}_{\lambda + \alpha}. \quad (4)$$

We often denote by  $\mathcal{L}_\lambda$  the  $\mathcal{L}_{\langle \lambda \rangle}$ .

## 2.3 Shifted conformal vector

We choose the (shifted) energy momentum field of  $\mathcal{L}$  in the form

$$T(z) = \frac{1}{2} \sum_{1 \leq i, j \leq l} c^{ij} : \alpha_i(z) \alpha_j(z) : + Q_0 \partial \rho(z) \quad (5)$$

where

$$Q_0 = \sqrt{p} - \frac{1}{\sqrt{p}}. \quad (6)$$

In other words, the conformal vector  $\omega \in \mathcal{F}_0 \subseteq \mathcal{L}$  is defined by

$$\omega = \frac{1}{2} \sum_{1 \leq i, j \leq l} c^{ij} (\alpha_i)_{-1} \alpha_j + Q_0 (\rho)_{-2} |0\rangle$$

and

$$\omega(z) = T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}.$$

The central charge  $c$  of  $T(z)$  is

$$c = l + 12(\rho, \rho)(2 - p - \frac{1}{p}) = l + h \dim \mathfrak{g}(2 - p - \frac{1}{p}) \quad (7)$$

and for  $\lambda \in \frac{1}{\sqrt{p}}Q'$ , the conformal weight  $\Delta_\lambda$  of  $|\lambda\rangle$  is

$$\Delta_\lambda = \frac{1}{2} |\lambda - Q_0 \rho|^2 + \frac{c-l}{24} = \frac{1}{2} |\lambda|^2 - Q_0(\lambda, \rho). \quad (8)$$

## 2.4 Screening operators and narrow screening operators

We consider the screening operators  $f_i$  and narrow screening operators  $F_i$  on  $\mathcal{L}_\lambda$  defined by

$$f_i = |\sqrt{p} \alpha_i\rangle_0, \quad (9)$$

$$F_i = | - \frac{1}{\sqrt{p}} \alpha_i \rangle_0, \quad (10)$$

for  $i = 1, \dots, l$ .

*Remark 1.* Strictly speaking, the definitions of  $F_i$  are different for each  $\mathcal{L}_\lambda$  and our definition of  $F_i$  is that in the case of  $\lambda \in \sqrt{p}Q'$  (see [CRW]), thus we have to denote not  $F_i$  but  $F_{i,\lambda}$  on  $\mathcal{L}_\lambda$ . However, the differences of definitions do not effect on the proof of our main results, and thus, we denote  $F_i$  by  $F_{i,\lambda}$  on  $\mathcal{L}_\lambda$  for simplicity.

A straightforward calculation gives the relations

$$[f_i, T(z)] = [F_i, T(z)] = 0 \quad (11)$$

and

$$[f_i, F_j] = 0. \quad (12)$$

In particular, (11) means that  $f_i$  and  $F_i$  preserve the conformal grading.

## 2.5 Definition of the logarithmic $W$ -algebras and results on the triplet $W$ -algebras

**Definition 2.** The sub VOA of  $\mathcal{L}$

$$W(p)_Q = \bigcap_{i=1}^l \ker F_i|_{\mathcal{L}} \subseteq \mathcal{L}.$$

called the logarithmic  $W$ -algebra associated to  $p$  and  $Q$ . In the case of  $A_1$  type,  $W(p)_Q$  is the triplet  $W$ -algebra (e.g. [AM1]).

If  $\mathfrak{g}$  is of  $A_1$  type, the following results are obtained.

**Theorem 1** (Adamovic-Milas, etc.).

1. Let  $L_k$  be the Virasoro VOA at level  $k = p - 2$ . Then,  $\ker f|_{\mathcal{F}_0} \simeq L_k$ .
2. Let  $\mathcal{R}_m$  be the  $\dim = m$  irreducible  $\mathfrak{sl}_2$ -module and  $L_{k,\mu}$  the irreducible  $L_k$ -module. Then,

$$\ker F|_{\mathcal{L}_\lambda} \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{R}_{2n+1+\bar{\lambda}} \otimes L_{k,-n\sqrt{p}\alpha_1+\lambda} \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \bigoplus_{i=0}^{2n+\bar{\lambda}} f^i L_k|_{-n\sqrt{p}\alpha_1+\lambda}.$$

3. The triplet  $W$ -algebra  $\ker F|_{\mathcal{L}}$  is simple  $C_2$ -cofinite and irrational VOA.
4.  $\{\ker F|_{\mathcal{L}_\lambda}\}$  is the complete set of the irreducible modules of the triplet  $W$ -algebra  $\ker F|_{\mathcal{L}}$ .

The author generalized Theorem 1.(1),(2) by using the geometric method introduced in [FT]. We give the setting and main results in the next section.

## 3 Feigin-Tipunin algebras and main results

### 3.1 Feigin-Tipunin algebras

For  $1 \leq i \leq l$ , we consider the following operators  $h_i$  acting on  $\mathcal{L}_\lambda$ :

$$h_i = -\frac{1}{\sqrt{p}}(\alpha_i)_0 + \frac{1}{\sqrt{p}}(\alpha_i, \bar{\lambda}), \quad (13)$$

where  $\bar{\lambda}$  is as in (3).

The operator  $h_i$  does not commute with  $T(z)$ , but sometimes we also call  $h_i$  the screening operator in this paper for simplicity.

**Theorem 2** ([FT, Theorem 4.1]).

1. The above operators  $\{f_i\}_{i=1}^l$  and  $\{h_i\}_{i=1}^l$  induces the action of  $\mathfrak{b}$  on  $\mathcal{L}_\lambda$ .
2. The action of  $\mathfrak{b}$  in (1) is integrable.

Theorem 2.(1) is proved in [FT] and the author proved Theorem 2.(2) in his master thesis.

For  $\lambda \in \Lambda$ , we consider the homogeneous vector bundle

$$\xi_\lambda = G \times_B \mathcal{L}_\lambda \quad (14)$$

on the flag variety  $G/B$ . The action of  $B$  on  $G$  is given by the right multiplication and on  $\mathcal{L}_\lambda$  by the one given in Theorem 1.

**Lemma 1.**  $H^0(\xi_0)$  has the structure of vertex operator algebra and  $H^0(\xi_\lambda)$  is a  $H^0(\xi_0)$ -module.

**Definition 3.** The VOA  $H^0(\xi_0)$  is called Feigin-Tipunin algebra associated to  $p \in \mathbb{Z}_{\geq 2}$  and  $Q$ .

Now we are ready to describe the main theorem.

### 3.2 Main results

**Theorem 3.** (Main theorem)

1. We have the isomorphism

$$H^n(\xi_\lambda) \simeq \begin{cases} 0 & (n \geq 1), \\ \bigcap_{i=1}^l \ker F_i|_{\mathcal{L}_\lambda} & (n = 0), \end{cases}$$

of modules of the vertex operator algebra  $\mathcal{WLX}_1(p) \simeq \bigcap_{i=1}^l \ker F_i|_{\mathcal{L}}$  (in the case of  $\lambda = 0$ ).

2. For  $\mu \in Q'_+$ , we denote by  $\mathcal{R}_\mu$  the finite dimensional highest weight irreducible  $\mathfrak{g}$ -module with highest weight  $\mu$  and  $\text{Weight}(\mathcal{R}_\mu)$  by the set of weights of  $\mathcal{R}_\mu$ . Let  $\mathcal{W}^k(\mathfrak{g})$  be the principal universal affine  $W$ -algebra ([Ar]) of level  $k = p - h$ .

Then, We have the free field realization of the  $W$ -algebra

$$\mathcal{W}^k(\mathfrak{g}) = \bigcap_{i=1}^l \ker f_i|_{\mathcal{F}_0}$$

and the  $G$ -module isomorphism

$$\begin{aligned} H^0(\xi_\lambda) &\simeq \bigoplus_{\alpha \in Q'_+ \cap Q} \mathcal{R}_{\alpha+\tilde{\lambda}} \otimes \mathcal{W}^k(\mathfrak{g})|_{-\sqrt{p}\alpha+\lambda} \\ &= \bigoplus_{\alpha \in Q'_+ \cap Q} \bigoplus_{\substack{1 \leq i_1, \dots, i_N \leq l \\ \alpha + \tilde{\lambda} - \sum_{j=1}^N \alpha_{i_j} \in \text{Weight } \mathcal{R}_{\alpha+\tilde{\lambda}}}} f_{i_1} \cdots f_{i_N} \mathcal{W}^k(\mathfrak{g})|_{-\sqrt{p}\alpha+\lambda} \subseteq \mathcal{L}_\lambda. \end{aligned}$$

In particular, we obtain the  $\mathcal{W}^k(\mathfrak{g})$ -module extension

$$\bigcap_{i=1}^l \ker F_i|_{\mathcal{L}_\lambda} = \bigoplus_{\alpha \in Q'_+ \cap Q} \bigoplus_{\substack{1 \leq i_1, \dots, i_N \leq l \\ \alpha + \tilde{\lambda} - \sum_{j=1}^N \alpha_{i_j} \in \text{Weight } \mathcal{R}_{\alpha+\tilde{\lambda}}}} f_{i_1} \cdots f_{i_N} \mathcal{W}^k(\mathfrak{g})|_{-\sqrt{p}\alpha+\lambda}.$$

3. We have the full character formulas

$$\text{Tr}_{\bigcap_{i=1}^l \ker F_i|_{\mathcal{L}_\lambda}} q^{L_0 - \frac{c}{24}} z_1^{\alpha_1} \cdots z_l^{\alpha_l} = \sum_{\alpha \in Q'_+ \cap Q} \chi_{\alpha+\tilde{\lambda}}^{\mathfrak{g}}(z) \left( \sum_{\sigma \in W} (-1)^{L(\sigma)} \frac{q^{\frac{1}{2}|\sqrt{p}\sigma(\alpha+\rho+\tilde{\lambda})-\tilde{\lambda}-\frac{1}{\sqrt{p}}\rho|^2}}{\eta(q)^l} \right),$$

where  $\chi_{\beta}^{\mathfrak{g}}(z)$  is the Weyl character formula of  $\mathcal{R}_{\beta}$  and  $L(\sigma)$  the length of  $\sigma \in W$ ,  $\eta(q) = (q; q)_{\infty} = \prod_{n \geq 1} (1 - q^n)$ .

**Remark 2.** Theorem 3.(1) claims that the Feigin-Tipunin algebras are geometric realizations of the logarithmic  $W$ -algebras. Theorem 3.(2) is generalization of Theorem 1.(1), (2). However, strictly speaking, we have to show that each  $\mathcal{W}^k(\mathfrak{g})$ -modules that appear in the modules of the logarithmic  $W$ -algebras  $\bigcap_{i=1}^l \ker F_i|_{\mathcal{L}_\lambda}$  is irreducible. At least in the case of  $p \geq h$  and  $\alpha = \lambda = 0$ , this claim is shown [CrM]. In other words, under the assumption of  $p \geq h$ ,  $\mathcal{W}^k(\mathfrak{g})$  is simple  $W$ -algebra.

## 参考文献

- [AM1] D. Adamovic and A. Milas. On the triplet vertex algebra  $W(p)$ , *Advances in Mathematics* 217 (2008), 2664-2699.
- [AM2] D. Adamovic and A. Milas. The  $N=1$  triplet vertex operator superalgebras, *Communications in Mathematical Physics*, 288 (2009), 225-270.
- [AM3] D. Adamovic and A. Milas. The structure of Zhu's algebras for certain  $W$ -algebras, *Advances in Mathematics*, 227 (2011) 2425-2456.
- [Ar] T. Arakawa. Representation theory of  $W$ -algebras, *Invent.Math.*, 169(2007), no.2, 219-320.
- [B] N. Bourbaki. Lie Groups and Lie Algebras. Chapter 4-6, *Elements of Mathematics* (Berlin), Springer-Verlag, Berlin, 2002, Translated from the 1968 French original by Andrew Pressly. MR 1890629
- [CrM] T. Creutzig, A. Milas. Higher rank partial and false theta functions and representation theory, *Adv.Math.*, 314(2017), no.9, 203-227.
- [CRW] T. Creutzig, D. Ridout, S. Wood. Coset Construction of Logarithmic  $(1, p)$  Models, *Letters in Mathematical Physics*, May(2014), Vol.104, 553-583.
- [D] C. Dong. Vertex algebras associated with even lattices, *J. Algebra* 161 (1993) 245-265.
- [FGST1] B. L. Feigin, A.M. Gainutdinov, A. M. Semikhatov, and I. Yu Tipunin. The Kazhdan-Lusztig correspondence for the representation category of the triplet  $W$ -algebra in logarithmic conformal field theories. (Russian) *Teoret. Mat. Fiz.* 148 (2006), no. 3, 398-427.
- [FGST2] B. L. Feigin, A.M. Gainutdinov, A. M. Semikhatov, and I. Yu Tipunin. Logarithmic extensions of minimal models: characters and modular transformations. *Nuclear Phys. B* 757 (2006), 303-343.
- [FGST3] B. L. Feigin, A. M. Gainutdinov, A. M. Semikhatov, and I. Yu Tipunin. Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center. *Comm. Math. Phys.* 265 (2006), 47-93.
- [FT] B. L. Feigin, I. Yu. Tipunin. Logarithmic CFTs connected with simple Lie algebras, arXiv:1002.5047.
- [NT] K. Nagatomo and A. Tsuchiya. The Triplet Vertex Operator Algebra  $W(p)$  and the Restricted Quantum Group at Root of Unity, Exploring new structures and natural constructions in mathematical physics, 149, *Adv. Stud. Pure Math.*, 61, Math. Soc. Japan, Tokyo, 2011., arXiv:0902.4607.
- [TW] A. Tsuchiya. S. Wood. On the extended  $W$ -algebra of type  $\mathfrak{sl}_2$  at positive rational level, *International Mathematics Research Notices*, Vol.2015, 14, 5357-5435.